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Statistical group theory and the distribution of angular momentum states: II

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Abstract. The statistical distribution of the angular momentum states J of a many-particle system j^N may be approximated by a Wigner-type distribution involving two parameters. It is shown that these two parameters may be determined group theoretically to yield simple expressions involving just j and N. A connection with the compositions of integers in number theory is noted.

1. Introduction

In this paper we extend an earlier study (Cleary and Wybourne 1971, hereafter referred to as I) of the statistical distribution of the angular momentum states J of a configuration j^N of N identical particles each of angular momentum j, a problem of relevance in atomic and nulcear physics. This problem is also relevant to discussions concerning the statistical behaviour of group-subgroup decompositions. Connections with the theory of compositions in number theory are noted.

Consider a semisimple Lie group G with a defining vector irrep 1_G which contains the three-dimensional rotation group SO_3 as a maximally embedded subgroup such that under $G\downarrow SO_3$

$$1_G\downarrow D^j \tag{1}$$

where D^j is the irrep of SO_3 of dimension $(2j+1)$ and j is the maximal weight of D^j. If j is an integer then G may be identified with SU_{2j+1} and SO_{2j+1} while if j is a half-integer with SU_{2j+1} and Sp_{2j+1}.

An arbitrary irrep λ_G of G may be decomposed under $G\downarrow SO_3$ as

$$\lambda_G\downarrow\sum_J g_{\lambda J}D^J \tag{2}$$

where $g_{\lambda J}$ is the number of times D^J appears in the decomposition. It has been observed in I that the numbers $g_{\lambda J}$, under certain conditions, tend to be distributed with respect to the maximal weights J of D^J according to a Wigner-type distribution

$$g_{\lambda J}\sim A(J+\tfrac{1}{2})\exp[-(J+\tfrac{1}{2})^2/2\sigma^2] \tag{3}$$

where A and σ depend on the group G, the partition (λ) labelling λ_G and the j and N of the configuration j^N.

In this paper we improve a number of the earlier results obtained in I and deduce explicit expressions for A and σ leading to a concise description of the statistical aspects of the relevant group-subgroup decompositions.

2. Angular momentum states and partitions

The characters of the irreps of SU_n, SO_n, Sp_n may all be expanded as sums of products of Schur functions $\{\lambda\}$ (Wybourne 1969, King 1975). Likewise any Schur function $\{\lambda\}$ may be expanded as a sum of products of Schur functions $\{k\}$ that involve just one part k where k is an integer. As shown in I the enumeration of the angular momentum states J of a configuration j^N requires the evaluation of plethysms (Littlewood 1958) of the type

$$[p]\otimes\{k\}=\sum_r g_{pkr}[r] \tag{4}$$

where p is an integer or half-integer and g_{pkr} is the coefficient of ρ^{-r} in the expansion of

$$\rho^{-pk}(1-\rho)\prod_{i=1}^{k}[(1-\rho^{2p+i})/(1-\rho^i)]. \tag{5}$$

Let $(1^n)_k(x)_Q$ be the number of ordered partitions of the integer x into n integer parts with no part exceeding k. Consideration of the relevant partition generating function leads to

$$g_{pkr}=\nabla(1^{2p})_k(x)_Q=(1^{2p})_k(x)_Q-(1^{2p})_k(x-1)_Q \tag{6}$$

with $x=pk-r$. Thus the distribution of the g_{pkr} with respect to r comes from the difference in the distributions of two sets of partition numbers leading as in I to the g_{pkr} being representable as a sum of Wigner distributions and under further approximation as a single dominant Wigner distribution.

The partition numbers $(1^n)_k(x)_Q$ may be expressed as a sum of P-type partition numbers introduced in I. Consider an ordered partition

$$(\lambda)=(\lambda_1^{m_1}\lambda_2^{m_2}\ldots\lambda_i^{m_i}\ldots)$$

where m_i is the number of times the part λ_i is repeated in the partition (λ).

A P-type partiton, of symmetry type $(\lambda)_k(x)$, is a partition of the integer x with no part exceeding k and at least λ_i parts equal. P-type partitions that differ by a distinct permutation of m_i sets of integers, each set involving λ_i equal parts, are said to constitute distinct P-type partitions. We let $(\lambda)_k(x)_P$ denote the number of P-type partitions of symmetry type (λ) of the integer x with no part exceeding k. Thus we find $(32^21)_3(8)_P=15$ since we have the 15 distinct P-type partitions of symmetry type (32^21):

(000) (00) (33) (2)	(000) (33) (00) (2)	(000) (33) (11) (0)
(000) (11) (33) (0)	(111) (11) (00) (3)	(111) (00) (11) (3)
(000) (22) (22) (0)	(222) (00) (00) (2)	(222) (11) (00) (0)
(222) (00) (11) (0)	(000) (22) (11) (2)	(000) (11) (22) (2)
(111) (22) (00) (1)	(111) (00) (22) (1)	(111) (11) (11) (1).

The numbers $(\lambda)_k(x)_P$ where (λ) involves q distinct parts λ_i may be generated by noting that $(\lambda)_k(x)_P$ is the coefficient of y^x in the expansion of

$$\prod_{i=1}^{q}\left(\sum_{d=0}^{k}y^{d\lambda_i}\right)^{m_i}. \tag{7}$$

Thus $(32^21)_3(8)_P$ is the coefficient of y^8 in the expansion of $(1+y^3+y^6+y^9)\times(1+y^2+y^4+y^6)^2(1+y+y^2+y^3)$ which is 15.

The numbers $(1^n)_k(x)_Q$ are related to the P-type partition numbers $(\lambda)_k(x)_P$ by the expansion

$$(1^n)_k(x)_Q = \frac{1}{n!}\sum_{\lambda\vdash n} h_\lambda(\lambda)_k(x)_P \tag{8}$$

where

$$h_\lambda = n!\left(\prod_i m_i!\, i^{m_i}\right)^{-1}. \tag{9}$$

Thus equation (6) may be rewritten as

$$g_{pkr} = \frac{1}{(2p)!}\sum_{\lambda\vdash 2p} h_\lambda \nabla(\lambda)_k(x)_P. \tag{10}$$

Use of (7) in (10) leads to rapid computer evaluation of the terms in (10) for large values of p and k and shows that the term $\nabla(1^{2p})_k(x)_P$ dominates and very closely approximates a Wigner distribution. The number $(1^{2p})_k(x)_P$ is simply the number of compositions of the integer x into n parts with no part exceeding k.

The asymptotic form of the number of compositions of an integer subject to various restrictions has been discussed in the mathematical literature (e.g. Szekeres 1953, Star 1975). The asymptotic form of $(1^n)_k(x)_P$ does not seem to have been considered. Our work suggests that it should be possible to show that $\nabla(1^n)_k(x)_P$ does asymptotically tend to the Wigner distribution.

3. Evaluation of A and σ

Equation (3) can be used to evaluate various summands to yield expressions for A and σ in terms of simply calculated group properties. Thus we have, as in I,

$$\sum_J g_{\lambda J} \approx A\int_0^\infty (J+\tfrac{1}{2})\exp[-(J+\tfrac{1}{2})^2/2\sigma^2]\,\mathrm{d}J = A\sigma^2 \tag{11a}$$

and likewise

$$\sum_J (2J+1)g_{\lambda J} = A\sigma^3(2\pi)^{1/2} \tag{11b}$$

$$\sum_J J(J+1)(2J+1)g_{\lambda J} = A\sigma^3(2\pi)^{1/2}(3\sigma^2-\tfrac{1}{4}). \tag{11c}$$

The LHS of (11) have simple group theoretical interpretations. Thus (11a) yields the sum of the multiplicities arising in (2), (11b) yields the dimension D_λ of the irrep (λ) of G while (11c) gives a quantity proportional to the second-order Dynkin index of the irrep (λ). Taking the ratio of (11c) to (11b) leads to the following expressions for σ^2 for the irrep λ:

$$\mathrm{SU}_{2j+1}\sigma^2\{\lambda\} = \frac{1}{12}\left[(2j+1)\sum_i \lambda_i(\lambda_i-2i+2j+2)-\left(\sum_i\lambda_i\right)^2+1\right] \tag{12a}$$

$$\mathrm{SO}_{2j+1}\sigma^2[\lambda] = \frac{1}{12}\left((2j+2)\sum_i \lambda_i(\lambda_i+2j-2i+1)+1\right) \tag{12b}$$

$$\mathrm{Sp}_{2j+1}\sigma^2\langle\lambda\rangle = \frac{1}{12}\left(2j\sum_i \lambda_i(\lambda_i+2j-2i+3)+1\right). \tag{12c}$$

The value of A may then be deduced from (11b) to give

$$A = D_\lambda / \sigma^3 (2\pi)^{1/2}. \tag{13}$$

Thus (12) and (13) together permit an evaluation of A and σ for the Wigner distribution given in (3). The ratio of (11b) to (11c) gives the expectation value $\langle 2J+1 \rangle$ for a given irrep λ under $G \downarrow SO_3$ as

$$\langle 2J+1 \rangle = \sigma (2\pi)^{1/2}. \tag{14}$$

Likewise, for a Wigner distribution we predict that $g_{\lambda J}$ attains its maximum value for

$$J_m \sim \sigma - \tfrac{1}{2} \tag{15}$$

with

$$g_{\lambda J_m} \sim A\sigma \, e^{-1/2}. \tag{16}$$

4. An example

Consider the antisymmetric states of N identical fermions each with an angular momentum j. Hermite's reciprocity principle (Wybourne 1969) may be used to show that the total angular momentum values, J, for the antisymmetric states of j^N are enumerated by the plethysm

$$[j] \otimes \{1^N\} = [(2j - N + 1)/2] \otimes \{N\} = \sum g_{\{1^N\}J} [J]. \tag{17}$$

We have for SU_{2j+1}

$$\sigma^2 \{1^N\} = \tfrac{1}{12} [N(2j+1)(2j-N+2) - N^2 + 1] \tag{18}$$

and

$$D_{\{1^N\}} = \frac{(2j+1)!}{N!(2j+1-N)!}. \tag{19}$$

Use of (18) and (19) in (13) permits the evaluation of A. Thus A and σ may be determined from a knowledge of just j and N.

The plethysms in (17) were evaluated for $j = 31/2$ with N up to 16 to give for each N exact values of $g_{\{1^N\}J}$. The case for $N = 16$ is typical. We find from (13), (18) and (19)

$$D_{\{1^{16}\}} = 601\,080\,390 \qquad \sigma = 26.535 \qquad A = 12\,835.315. \tag{20}$$

The above numbers may be used in (15), (16) and (11a) and the calculated numbers compared with the exact numbers found from (17). We find

	J_m	$g_{\{1^{16}\}J_m}$	$\sum g_{\{1^{16}\}J}$
calculated	26.03	206 575	9037 425
exact	26	200 616	8908 546

with agreement to ~3%. These results show that even a single Wigner distribution is capable of closely approximating the exact calculations and could be useful in discussing the properties of many-particle systems involving high angular momentum.

5. Conclusion

We have succeeded in expressing the parameters of the Wigner distribution in terms of simple group theoretically determined quantities. A connection with the theory of the compositions of integers has been noted.

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